

Remarks on the Bergmann–Thomson Expression on Angular Momentum in General Relativity

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(February 7, 2008)

Abstract

Often it is asserted that only by using of the symmetric Landau–Lifschitz energy–momentum complex one is able to formulate a conserved angular momentum complex in General Relativity (**GR**). Obviously, it is an uncorrect statement. For example, years ago, Bergmann and Thomson have given other, very useful expression on angular momentum. This expression is closely joined to the non–symmetric, Einstein canonical energy–momentum complex. In the paper we review the Bergmann–Thomson angular momentum complex and compare it with that of given by Landau and Lifschitz.

04.20.Me.04.30.+x

I. BERGMANN–THOMSON EXPRESSION ON ANGULAR MOMENTUM IN GR

One can most easily obtain the canonical Bergmann–Thomson expression on total angular momentum density, matter and gravitation, in the following way. At first, let us transform the Einstein equations written in mixed form and multiplied by $\sqrt{|g|}$

$$\sqrt{|g|}G_i^k = \beta\sqrt{|g|}T_i^k \quad (1)$$

to the so-called *superpotential form*

$${}_EK_i^k = {}_FU_i^{[kl]}_{,l}, \quad (2)$$

where

$${}_EK_i^k := \sqrt{|g|}(T_i^k + {}_E t_i^k) \quad (3)$$

is the *canonical, Einstein's energy–momentum complex*, matter and gravitation, and ${}_FU_i^{[kl]}$ mean *von Freud superpotentials*.

From (2) we get, after series operations, the *Bergmann–Thomson energy–momentum complex* ${}_{BT}K^{jk}$, matter and gravitation. At first, we form

$$g^{ij}{}_EK_i^k = g^{ij}{}_FU_i^{[kl]}_{,l} = (g^{ij}{}_FU_i^{[kl]})_{,l} - {}_FU_i^{[kl]}g^{ij}_{,l} \quad (4)$$

or

$$g^{ij}{}_EK_i^k + {}_FU_i^{[kl]}g^{ij}_{,l} = {}_FU^{j[kl]}_{,l}. \quad (5)$$

Then, we write (5) in the form

$${}_{BT}K^{jk} = {}_FU^{j[kl]}_{,l}, \quad (6)$$

where

$${}_{BT}K^{jk} := {}_E K^{jk} + {}_FU_i^{[kl]}g^{ij}_{,l} =: \sqrt{|g|}(T^{jk} + {}_{BT}t^{jk}) \quad (7)$$

is the *Bergmann–Thomson energy–momentum complex* [1,2,3,4] of matter and gravitation which satisfies local conservation laws

$${}_{BT}K^{jk},_k = 0. \quad (8)$$

Here ${}_{BT}t^{jk} \neq {}_{BT}t^{kj}$ mean the components of the so-called *Bergmann–Thomson energy-momentum pseudotensor* of the gravitational field [1,2,3,4]. Finally, from (6) we get

$$\begin{aligned} x^i {}_{BT}K^{jk} - x^j {}_{BT}K^{ik} &= x^i {}_F U^{j[kl]},_l - x^j {}_F U^{i[kl]},_l \\ &= (x^i {}_F U^{j[kl]} - x^j {}_F U^{j[kl]})_{,l} - {}_F U^{j[ki]} + {}_F U^{i[kj]}, \end{aligned} \quad (9)$$

or

$$x^i {}_{BT}K^{jk} - x^j {}_{BT}K^{ik} + {}_{BT}S^{ijk} = M^{[ij][kl]},_l, \quad (10)$$

where

$$\begin{aligned} {}_{BT}S^{ijk} &:= {}_F U^{i[jk]} - {}_F U^{j[ik]} = \frac{\alpha}{\sqrt{|g|}} \left[(-g)(g^{kj}g^{il} - g^{ki}g^{jl}) \right]_{,l} \\ &=: \frac{\alpha}{\sqrt{|g|}} g^{[ij][kl]},_l \end{aligned} \quad (11)$$

and

$$M^{[ij][kl]} := x^i {}_F U^{j[kl]} - x^j {}_F U^{i[kl]}. \quad (12)$$

The expression

$$x^i {}_{BT}K^{jk} - x^j {}_{BT}K^{ik} + {}_{BT}S^{ijk} =: {}_{BT}M^{ijk} \quad (13)$$

is the *Bergmann–Thomson angular momentum complex* matter and gravitation and the quantities $M^{[ij][kl]}$ are *superpotentials* [1].

The complex ${}_{BT}M^{ijk}$ satisfies local conservation laws

$${}_{BT}M^{ijk},_k = 0. \quad (14)$$

One can interpret physically the angular momentum complex (13) as a sum of the *orbital part*

$$O^{ijk} := x^i {}_{BT}K^{jk} - x^j {}_{BT}K^{ik} = \sqrt{|g|}(x^i T^{jk} - x^j T^{ik}) + \sqrt{|g|}(x^i {}_{BT}t^{jk} - x^j {}_{BT}t^{ik}) \quad (15)$$

of the angular momentum density of matter and gravitation (The matter part includes also spin density [1]) and a *spinorial part*

$${}_{BT}S^{ijk} =_F U^{i[jk]} -_F U^{j[ik]} = \frac{\alpha}{\sqrt{|g|}} g^{[ij][kl]} ,_l \quad (16)$$

of the gravitational angular momentum density.

We have from (6)

$$2 {}_{BT}K^{[ij]} = \sqrt{|g|} ({}_{BT}t^{ij} - {}_{BT}t^{ji}) = {}_{BT}S^{ijk} ,_k = \left(\frac{\alpha}{\sqrt{|g|}} \right) ,_k g^{[ij][kl]} ,_l, \quad (17)$$

because the dynamical energy-momentum tensor of matter T^{ik} is symmetric: $T^{ik} = T^{ki}$.

The equality (17) justifies the above proposal of the physical interpretation of the pseudo-tensor ${}_{BT}S^{ijk} = (-) {}_{BT}S^{jik}$ as a quantity describing *canonical spin density* of the gravitational field ¹.

It is very interesting that formal application of the special-relativistic symmetrization procedure given by Belinfante (see, e.g., [5,6]) to the complex ${}_{BT}K^{ij}$ with $S^{ijk} = {}_{BT}S^{ijk}$ leads us immediately to the new, symmetric complex ${}_{BT}K^{Nij} = \sqrt{|g|} (T^{ij} - \frac{c^4}{8\pi G} G^{ij})$ which trivially vanishes, i.e., it leads us to the Lorentz and Levi-Civita solution of the energy-momentum problem for gravitational field: ${}_G T^{ik} = (-) \frac{c^4}{8\pi G} G^{ik}$.

From (10) we get the following expression on the components $M^{ik} = (-)M^{ki}$ of the global angular momentum of matter and gravitation for an isolated system endowed with an asymptotically Lorentzian coordinates ²

¹In Special Relativity the antisymmetric part of an energy-momentum tensor is proportional to the ordinary divergence of a quantity which describes canonical spin densities [5,6]. Here we follow this line.

²The Bergmann-Thomson expression (10), likely as the canonical Einstein energy-momentum complex, can be reasonably used only in the case of an isolated system endowed with an asymptotically flat coordinates (see, e.g., [2,3]).

$${}_{BT}M^{ik} = \frac{1}{c} \oint_{\text{over sphere having } R \rightarrow \infty} (x^i {}_F U^{j[0\alpha]} - x^j {}_F U^{i[0\alpha]}) n_\alpha r^2 d\Omega. \quad (18)$$

$r^2 = x^2 + y^2 + z^2$; n_α are the components of the unit (exterior) normal to the sphere and $d\Omega = \sin \theta d\theta d\varphi$.

For the Schwarzschild spacetime equipped with an asymptotically Lorentzian coordinates ($x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$) we get from (18) the expected result

$${}_{BT}M^{ik} = 0. \quad (19)$$

On the other hand, in the case of the stationary and axially symmetric Kerr's spacetime (see, e.g., [7,8,9]) we obtain, in an asymptotically Lorentzian coordinates also, that the only one component

$${}_{BT}M^{12} = (-) {}_{BT}M^{21} = mac - \frac{1}{3}mac = \frac{2}{3}mac \quad (20)$$

of the ${}_{BT}M^{ik}$ is different from zero.

One should interpret the result (20) as giving the *global angular momentum*, spinorial and orbital of matter and gravitation, for Kerr spacetime.

II. LANDAU–LIFSHITZ EXPRESSION ON ANGULAR MOMENTUM DENSITY IN GENERAL RELATIVITY AND ITS COMPARISON WITH BERGMANN–THOMSON EXPRESSION

One can easily obtain the Landau–Lifschitz expression on angular momentum density for matter and gravitation in the following way [9]. At first, one should transform the Einstein equations multiplied by $(-g)$

$$(-g)G^{ik} = \beta(-g)T^{ik} \quad (21)$$

to the *superpotential form*

$$(-g)(T^{ik} + {}_{LL}t^{ik}) = h^{ikl}_{,l}, \quad (22)$$

where

$$h^{ikl} = (-)h^{ilk} = \lambda^{iklm}_{\quad ,m} \quad (23)$$

and

$$\lambda^{iklm} := \alpha(-g)(g^{ik}g^{lm} - g^{il}g^{km}). \quad (24)$$

$$\alpha = \frac{1}{2\beta} = \frac{c^4}{16\pi G}.$$

t_{LL}^{ik} are components of the so-called *Landau–Lifschitz gravitational energy-momentum pseudotensor* and the sum

$$(-g)(T^{ik} + t_{LL}^{ik}) =:_{LL} K^{ik} \quad (25)$$

forms the so-called *Landau–Lifschitz energy-momentum complex*³ of matter and gravitation. $h^{ikl} = (-)h^{ilk}$ form the *Landau–Lifschitz superpotentials*.

Then, from the (22)–(23), one can obtain easily

$$\begin{aligned} x^i_{LL} K^{kl} - x^k_{LL} K^{il} &= x^i h^{klm}_{\quad ,m} - x^k h^{ilm}_{\quad ,m} \\ &= (x^i h^{klm} - x^k h^{ilm})_{,m} - h^{kli} + h^{ilk}, \end{aligned} \quad (26)$$

or

$$x^i_{LL} K^{kl} - x^k_{LL} K^{il} + _{LL} S^{ikl} = L^{[ik][lm]}_{\quad ,m}, \quad (27)$$

where

$$_{LL} S^{ikl} := h^{ikl} - h^{kil} \neq 0 \quad (28)$$

and

$$L^{[ik][lm]} := x^i h^{klm} - x^k h^{ilm}. \quad (29)$$

³The complex (25) has worse transformational properties than the Einstein canonical energy-momentum complex [2,3].

The expression (27) *exactly corresponds* to the Bergmann–Thomson expression (10). Namely, $x^i {}_{LL} K^{kl} - x^k {}_{LL} K^{il}$ gives *orbital part* of the angular momentum density of matter and gravitation, ${}^4 {}_{LL} S^{ikl} = h^{ikl} - h^{kil}$ gives the non–vanishing *spinorial part* of the gravitational angular momentum density and $L^{[ik][lm]} = x^i h^{klm} - x^k h^{ilm}$ are *superpotentials* for the total angular momentum density of matter and gravitation.

The total angular momentum density

$${}_{LL} M^{ikl} := x^i {}_{LL} K^{kl} - x^k {}_{LL} K^{il} + {}_{LL} S^{ikl} \quad (30)$$

satisfies local conservation laws

$${}_{LL} M^{ikl},_l = 0. \quad (31)$$

From (27) one can obtain the following integral expression on the global angular momentum of an isolated system endowed with an asymptotically Lorentzian coordinates ($x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$)

$${}_{LL} M^{ik} = \frac{1}{c} \oint_{\text{over sphere } S^2 \text{ having radius going to } \infty} L^{[ik][0\alpha]} n_\alpha r^2 d\Omega. \quad (32)$$

This expression gives the same values of the components $M^{ik} = (-)M^{ki}$ as the Bergmann–Thomson expression (18) gives. Especially, for the Kerr’s spacetime it gives

$${}_{LL} M^{12} = (-) {}_{LL} M^{21} = mac - \frac{1}{3} mac = \frac{2}{3} mac; \quad (33)$$

other components vanish.

However, in the case we have

$${}_{LL} S^{ikl} = \lambda^{klmi},_m \quad (34)$$

from which it follows

$${}_{LL} S^{ikl},_l = 0. \quad (35)$$

⁴Material part $(-g)(x^i T^{kl} - x^k T^{il})$ includes material spin also.

The last equality guarantees symmetry of the Landau–Lifschitz energy–momentum complex:

$${}_{LL}K^{ik} - {}_{LL}K^{ki} = S^{ikl}_{,l} = 0. \quad (36)$$

Landau–Lifschitz, in their book [9], modify the expression (32) on global angular momentum of an isolated system equipped with an asymptotically Lorentzian coordinates by using (34). Namely, they *subtract* the gravitational spin angular density ${}_{LL}S^{ikl} = (-){}_{LL}S^{kil}$ from the both sides of the (27) and obtain the *new expression* on angular momentum density of matter and gravitation

$$x^i {}_{LL}K^{kl} - x^k {}_{LL}K^{il} = (L^{[ik][lm]} - \lambda^{klmi})_{,m}. \quad (37)$$

This new expression includes the *total matter angular momentum density*, spinorial and orbital, but its gravitational part consists of the *orbital gravitational angular momentum density* only.

The new expression (37) leads to the following integrals on global quantities of an isolated system equipped with an asymptotically Lorentzian coordinates ($x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$)

$$M^{ik} = \frac{1}{c} \oint_{\text{over sphere } S^2 \text{ having radius } R \rightarrow \infty} (L^{[ik][0\alpha]} + \lambda^{i0\alpha k}) n_\alpha r^2 d\Omega. \quad (38)$$

In the case of the Schwarzschild spacetime equipped with an asymptotically Lorentzian coordinates ($x^0 = ct, x, y, z$) the expression (38) gives $M^{ik} = (-)M^{ki} = 0$ and in the case of the Kerr's spacetime it gives

$$M^{12} = (-)M^{21} = mac - \frac{1}{3}mac + \frac{1}{3}mac = mac; \quad (39)$$

other components of the M^{ik} vanish.

An analogical modification one can do with the Bergmann–Thomson expression (18). Namely, by using (11), one can rewrite (10) in the following form

$$x^i {}_{BT} K^{jk} - x^j {}_{BT} K^{ik} - \left(\frac{\alpha}{\sqrt{|g|}} \right)_{,l} g^{[ij][kl]} + \left(\frac{\alpha g^{[ij][kl]}}{\sqrt{|g|}} \right)_{,l} = M^{[ij][kl]}_{,l}. \quad (40)$$

By moving the term $(\frac{\alpha g^{[ij][kl]}}{\sqrt{|g|}})_{,l}$ onto the right hand side of the (40) one gets a new form of the Bergmann–Thomson expression (10) with the *new angular momentum complex* ${}_{BT} \overline{M}^{ikl} = (-) {}_{BT} \overline{M}^{kil}$ and with the *new superpotentials* $\overline{M}^{[ij][kl]}$.

Namely, we get

$$\begin{aligned} x^i {}_{BT} K^{jk} - x^j {}_{BT} K^{ik} - \left(\frac{\alpha}{\sqrt{|g|}} \right)_{,l} g^{[ij][kl]} \\ = \left(M^{[ij][kl]} - \frac{\alpha}{\sqrt{|g|}} g^{[ij][kl]} \right)_{,l} =: \overline{M}^{[ij][kl]}_{,l}. \end{aligned} \quad (41)$$

The last expression leads us to the following integrals on global quantities $\overline{M}^{ik} = (-) \overline{M}^{ki}$ in the case of an isolated system equipped with an asymptotically Lorentzian coordinates ($x^0 = ct, x, y, z$)

$$\begin{aligned} \overline{M}^{ik} &= \frac{1}{c} \oint_{\text{over sphere } S^2 \text{ having radius } R \rightarrow \infty} \left(M^{[ij][0\alpha]} - \alpha \frac{g^{[ij][0\alpha]}}{\sqrt{|g|}} \right) n_\alpha r^2 d\Omega \\ &:= \frac{1}{c} \oint_{\text{over sphere } S^2 \text{ having radius } R \rightarrow \infty} \overline{M}^{[ij][0\alpha]} n_\alpha r^2 d\Omega. \end{aligned} \quad (42)$$

The integrals (42) correspond to the Landau–Lifschitz integrals (38) and, especially, for the Kerr’s spacetime they give the same values as the Landau–Lifschitz integrals (38) give, i.e., they give

$$\overline{M}^{12} = (-) \overline{M}^{21} = mac - \frac{1}{3} mac + \frac{1}{3} mac = mac; \quad (43)$$

other components vanish.

So, the Bergmann–Thomson and Landau–Lifschitz integral expressions on global angular momentum are *fully equivalent* in the case of an isolated system equipped with an asymptotically Lorentzian coordinates. But we favorize the Bergmann–Thomson expression on angular momentum density because it has *better local transformational properties* in comparison with the Landau–Lifschitz expression and it is *more closely related* to the canonical energy–momentum complex ${}_E K_i^k = \sqrt{|g|} (T_i^k + {}_E t_i^k)$.

In the case of the Kerr's spacetime one should interpret the value $M^{12} = (-)M^{21} = \overline{M}^{12} = (-)\overline{M}^{21} = mac$ of the integrals (38) or (42) as *referring only to the material part of the angular momentum complex*. It is justified by the following fact [10]: the integral including the energy-momentum tensor of matter $T^{ik} = T^{ki}$ only

$$\begin{aligned} M_2^1 &= (-)M^{12} = \frac{1}{c} \int_{x^0=ct=\text{const}} T_i{}^0 \xi^i \sqrt{|g|} dx dy dz \\ &= \frac{c^3}{16\pi G} \oint_{\substack{\text{over sphere } S^2 \text{ having radius } R \rightarrow \infty}} \left(\nabla^0 \xi^\alpha - \nabla^\alpha \xi^0 \right) \sqrt{|g|} n_\alpha r^2 d\Omega, \end{aligned} \quad (44)$$

where $\xi^i = x\delta_2^i - y\delta_1^i$ are the components of the spatial Killing vector field $\vec{\xi} = \xi^i \partial_i$ which exists in the case, gives

$$M_2^1 = (-)M^{12} = (-)mac. \quad (45)$$

So, the *matter integral* (44) has the same value in the case as the integrals (38) and (42). In consequence, we conclude that only the material parts $(-)g(x^i T^{kl} - x^k T^{il})$, $\sqrt{|g|}(x^i T^{kl} - x^k T^{il})$ of the suitable angular momentum complex (Landau-Lifschitz or Bergmann-Thomson) give contribution to the integrals (38) and (42). Gravitational parts *give no contribution* in the used coordinates.

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